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Best Approximations of Alternating Series

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1. INTRODUCTION

There are several results in the literature on the problem of accelerating the convergence of alternating series. See, for example [3–9, 12, 14–19, 21, 22, 24]. Knopp's book [18] contains a chapter on the problem and Brezinski's monograph [8] contains many further references. Much emphasis has been given to what can be done with one series. Here we shall state a problem in a way that encompasses a wide class of alternating series, and obtain the best possible results for this class.

We consider the class of alternating series, $\sum_{n=0}^{\infty}(-1)^n a_n$, where $\{a_n\}$ is a moment sequence: that is, when $a_n = \int_0^1 t^n \, d\phi(t)$, where $\int_0^1 |d\phi(t)| < \infty$ (or $\int_0^1 |d\phi(t)| \leqslant 1$). We shall also consider subclasses in which $a_n = \int_0^1 t^n \psi(t)\, dt$ with $\psi \in L^q$ $(1 \leqslant q \leqslant \infty)$. We permit ϕ (and ψ) to be complex valued.

Our main class includes those series obtained from totally monotone sequences $\{a_n\}$: that is, whenever $\Delta^k a_n \geqslant 0$ for all $n \geqslant 0$ and all $k \geqslant 0$, where $\Delta^0 a_n = a_n$, $\Delta^1 a_n = a_n - a_{n+1}$ and $\Delta^k a_n = \Delta^1 \ (\Delta^{k-1} a_n)$. Such series are considered by Bilodeau [4] and Grosjean [12].

Brezinski [8] defines a sequence $\{s_n\}$ to be totally oscillating if the sequence $\{(-1)^n s_n\}$ is totally monotone. He also shows that a convergent

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totally oscillating sequence must converge to zero. Our class includes those convergent alternating series whose sequence of partial sums $\{s_n\}$ satisfies the condition "$\{s_n - s\}$ is totally oscillating," where s is the sum of the series.

For those alternating series which are convergent, we ask the question:

> "*What is the best approximation to the sum that can be obtained from the first $n+1$ terms of a series in the class?*"

This leads naturally to consideration of triangular matrix transformations and, indeed, the majority of the results in the literature are of this type. These results have all concentrated on sequence-to-sequence matrix transformations. We have discovered that it is better to consider first a series-to-sequence matrix transformation, answer the question, and then deduce the corresponding, and more usually employed, sequence-to-sequence matrix.

For the remainder of this section, we give the notations to be used throughout this paper.

Let $C = (c_{n,k})$ be a series-to-sequence triangular matrix (summability method). Thus $c_{n,k} = 0$ whenever $k > n$. Define

$$\sigma_n = \sum_{k=0}^{n} c_{n,k}(-1)^k a_k \qquad \text{and} \qquad \gamma_n(t) = \sum_{k=0}^{n} c_{n,k} t^k.$$

Thus

$$\sigma_n = \int_0^1 \sum_{k=0}^{n} c_{n,k}(-1)^k t^k \, d\phi(t) = \int_0^1 \gamma_n(-t)\, d\phi(t).$$

Also

$$\sum_{k=0}^{n} (-1)^k a_k = \int_0^1 \sum_{k=0}^{n} (-1)^k t^k \, d\phi(t) = \int_0^1 (1+t)^{-1}\, d\phi(t) + r_n,$$

where

$$r_n = (-1)^n \int_0^1 t^{n+1}(1+t)^{-1}\, d\phi(t).$$

Whenever the series $\sum_{k=0}^{\infty}(-1)^k a_k$ is convergent, we have that ϕ is continuous at 1, and so that $r_n \to 0$ as $n \to \infty$, yielding that

$$\sum_{k=0}^{\infty} (-1)^k a_k = \int_0^1 (1+t)^{-1}\, d\phi(t).$$

Define

$$s = \int_0^1 (1+t)^{-1}\, d\phi(t),$$

so that

$$\sigma_n - s = \int_0^1 \{\gamma_n(-t) - (1+t)^{-1}\}\, d\phi(t),$$

and so

$$|\sigma_n - s| \leqslant \max_{0 \leqslant t \leqslant 1} |\gamma_n(-t) - (1+t)^{-1}| \cdot \int_0^1 |d\phi(t)|.$$

We consider, without loss of generality, the class in which $\int_0^1 |\psi(t)|\, dt \leqslant 1$. By taking the supremum over this class, we obtain that

$$\sup |\sigma_n - s| = \max_{0 \leqslant t \leqslant 1} |\gamma_n(-t) - (1+t)^{-1}| = \|\gamma_n(-t) - (1+t)^{-1}\|_\infty .$$

For the subclasses given by $\int_0^1 |\psi(t)|^q\, dt \leqslant 1$, we have

$$\sup |\sigma_n - s| = \left(\int_0^1 |\gamma_n(-t) - (1+t)^{-1}|^p\, dt\right)^{1/p} = \|\gamma_n(-t) - (1+t)^{-1}\|_p,$$

where $1/p + 1/q = 1$.

Thus we define the error for the row of order n of the matrix C, operating on the first $n+1$ terms of the series in our class to be

$$\varepsilon_n^{(p)}(C) = \|\gamma_n(-t) - (1+t)^{-1}\|_p \qquad (1 \leqslant p \leqslant \infty).$$

For each n, the best approximation occurs when this error is minimal: that is, we minimize $\varepsilon_n^{(p)}$ by letting γ_n vary over the class of polynomials of degree n.

Let $B = (b_{n,k})$ be the sequence-to-sequence matrix (summability method) that corresponds to the series-to-sequence matrix $C = (c_{n,k})$ given above. Thus

$$\sigma_n = \sum_{k=0}^{n} b_{n,k} s_k = \sum_{k=0}^{n} b_{n,k} \sum_{j=0}^{k} (-1)^j a_j = \sum_{j=0}^{n} (-1)^j a_j \sum_{k=j}^{n} b_{n,k},$$

so that

$$c_{n,k} = \sum_{j=k}^{n} b_{n,j}.$$

Define $\beta_n(t) = \sum_{k=0}^{n} b_{n,k} t^k$. It now follows that

$$(1-t)\,\gamma_n(t) = \gamma_n(0) - t\beta_n(t),$$

where $\gamma_n(0) = \beta_n(1)$ is the row sum of the row of order n of the sequence-to-sequence matrix B.

In the sections which follow, we obtain the best approximations in the cases $p = \infty$ and $p = 2$. In a forthcoming paper, Fiedler and Jurkat [11] obtain the best approximation in the case $p = 1$. In all cases, it will be shown that

$$\varepsilon_n^{(p)} \sim c_p \lambda^n,$$

where $\lambda = 3 - 2\sqrt{2}$ and c_p is a constant depending only on p. We show that $c_\infty = 1/4$ and $c_2 = \lambda \sqrt{\pi}/2$. Fiedler and Jurkat show that $c_1 = 4\lambda^2$. Note that

$$c_\infty = 0.25 > c_2 \doteq 0.152 > c_1 \doteq 0.118.$$

Since $\varepsilon_n^{(p)}$ is an increasing function of p, it follows that the quotient $\varepsilon_n^{(p)}/\lambda^n$ varies within fixed positive bounds. However, as of now, neither the asymptotic for the error nor the corresponding best matrices are known for $p \notin \{1, 2, \infty\}$.

2. Solution to the Problem when $p = \infty$

We have

$$\varepsilon_n^{(\infty)} = \|\gamma_n(-t) - (1+t)^{-1}\|_\infty.$$

In order to minimize this expression as γ_n varies over the class of polynomials of degree n, we must find the best L^∞ polynomial approximation to $(1+t)^{-1}$ over the interval $[0, 1]$. The solution to this problem is known, due to Čebyšev [26] and can be found, for example, in Akhieser's book [2]. Thus we can calculate each row of the corresponding matrices C and B, and the minimal value of $\varepsilon_n^{(\infty)}$. It turns out that these matrices are particularly nice and can be determined explicitly. They are, in fact, regular summability methods. We shall also investigate (in Section 5) the asymptotic behaviour of the matrix B and find the Gaussian distribution to which it is asymptotic.

Lemma 2.1 (Čebyšev). *The best L^∞ polynomial approximation to $(t-a)^{-1}$, with $a > 1$, over the interval $[-1, 1]$ is given by*

$$p_n(t) = (t-a)^{-1} - \Phi_n(t),$$

where

$$\Phi_n(t) = \frac{M}{2}\left\{v^n\left(\frac{a-v}{1-av}\right) + v^{-n}\left(\frac{1-av}{a-v}\right)\right\},$$

with

$$t = \tfrac{1}{2}(v + v^{-1}), \qquad a = \tfrac{1}{2}(\alpha + \alpha^{-1}), \qquad \alpha = a - (a^2 - 1)^{1/2} < 1$$

and

$$M = 4\alpha^{n+2}(1 - \alpha^2)^{-2}.$$

Further, the error in this approximation is

$$M = \max_{-1 \leqslant t \leqslant 1} |(t - a)^{-1} - p_n(t)|.$$

In our problem, $a = 3$, $\alpha = 3 - \sqrt{8}$: we then set $t = 1 - 2x$ and multiply the resulting expression by -2 to obtain $\gamma_n(-x)$, and, *a fortiori*, the row of order n of the matrix C. The error, in our case, is then $M/2$.

For the rest of this paper, let $\lambda = 3 - \sqrt{8}$. Observe then that

$$\varepsilon_n^{(\infty)}(C) = \lambda^n/4.$$

We also have that

$$\begin{aligned}\Phi_0(t) &= \frac{1}{16}\{(\lambda - v)^2 + (1 - \lambda v)^2\}/\{(1 - \lambda v)(\lambda - v)\}\\ &= \frac{1}{16}\{(v + v^{-1})(\lambda + \lambda^{-1}) - 4\}/\{v + v^{-1} - \lambda - \lambda^{-1}\}\\ &= \frac{1}{8}(at - 1)/(t - a) = \frac{3}{8} + \frac{1}{t - 3}.\end{aligned}$$

Thus $\gamma_0(-x) = \frac{3}{4} = \gamma_0(x)$.

For $n \geqslant 1$ we have

$$\begin{aligned}&v^n\left(\frac{\lambda - v}{1 - \lambda v}\right) + v^{-n}\left(\frac{1 - \lambda v}{\lambda - v}\right)\\ &\quad = \{(v^{n+1} + v^{-n-1}) - 2\lambda(v^n + v^{-n}) + \lambda^2(v^{n-1} + v^{-n+1})\}/\{2\lambda(t - 3)\}.\end{aligned}$$

We define $W_{n+1}(t) = \{(v^{n+1} + v^{-n-1}) - 2\lambda(v^n + v^{-n}) + \lambda^2(v^{n-1} + v^{-n+1})\}/2$. Recalling that the Čebyšev polynomial of degree n (the best L^∞ polynomial approximation to x^{n+1} over $[-1, 1]$) is given by $T_n(t) = (v^n + v^{-n})/2$, we obtain that

$$W_{n+1}(t) = T_{n+1}(t) - 2\lambda T_n(t) + \lambda^2 T_{n-1}(t).$$

It follows that W_{n+1} is a polynomial of degree $n + 1$. Thus

$$\Phi_n(t) = \lambda^{n-1} W_{n+1}(t)/\{16(t - 3)\}$$

and so

$$-\gamma_n(x) = \{16 - \lambda^{n-1} W_{n+1}(1+2x)\}/\{16(x-1)\}.$$

Since it is known [20] that

$$T_n(1+2x) = (-1)^n T_{2n}(\sqrt{-x})$$

and

$$T_{2n}(t) = \sum_{k=0}^{n} 2^{2(n-k)-1}(-1)^k \left\{2\binom{2n-k}{k} - \binom{2n-k-1}{k}\right\} t^{2n-2k},$$

we deduce that

$$T_n(1+2x) = \sum_{k=0}^{n} \frac{n(n+k-1)!}{(n-k)!\,2k!}(4x)^k,$$

and so we can obtain $W_{n+1}(1+2x)$ and $\gamma_n(x)$ explicitly.

However, it is more convenient to obtain the rows of the corresponding sequence-to-sequence matrix B. We observe that

$$(1-x)\,\gamma_n(x) = \gamma_n(0) - x\beta_n(x) = 1 - \lambda^{n-1} W_{n+1}(1+2x)/16.$$

It now follows, for $0 \leqslant k \leqslant n$, that

$$\begin{aligned} b_{n,k} &= \frac{\lambda^{n-1}2^{2k-3}}{k+1}\left\{(n+1)\binom{n+k+1}{n-k}\right. \\ &\qquad \left. - 2\lambda n\binom{n+k}{n-k-1} + \lambda^2(n-1)\binom{n+k-1}{n-k-2}\right\} \\ &= \frac{\lambda^{n-1}2^{2k-2}(n+k-1)!}{(n-k)!\,(2k+2)!}\lambda_{n,k}, \end{aligned}$$

where

$$\begin{aligned} \lambda_{n,k} &= (n+1)(n+k+1)(n+k) \\ &\qquad - 2\lambda n(n+k)(n-k) + \lambda^2(n-1)(n-k)(n-k-1) \\ &= \lambda\{4\sqrt{2}(\sqrt{2}n+1)\,k^2 + 2(4\sqrt{2}n^2 + 9n + 2\sqrt{2})\,k \\ &\qquad + 2n(2n^2 + 4\sqrt{2}n + 3)\}. \end{aligned}$$

(Binomial coefficients $\binom{\mu}{\nu}$ with $\mu < \nu$ have, of course, value 0.)

Thus $b_{n,k} > 0$ whenever $0 \leqslant k \leqslant n$ so that B is a positive triangular matrix.

We also observe that

$$\sum_{k=0}^{n} b_{n,k} = \gamma_n(0) = 1 - \frac{\lambda^{n-1} W_{n+1}(1)}{16}$$
$$= 1 - \frac{\lambda^{n-1}(1-\lambda)^2}{16}$$
$$= 1 - \frac{\lambda^n}{4} \to 1 \qquad \text{as } n \to \infty \text{ since } 0 < \lambda < 1.$$

Further, an application of Stirling's formula yields that $\lim_{n\to\infty} b_{n,k} = 0$ for each k.

It now follows from Toeplitz's theorem (see, for example [13, p. 43]) that B is a regular matrix (that is, it transforms convergent sequences into convergent sequences and preserves the value of the limit).

3. Approximation Results in L^2

In this section, we develop results concerning the best L^2 polynomial approximation to $(t-a)^{-1}$ over the interval $[-1, 1]$, where $a > 1$, so that we have a result analogous to Čebyšev's (Lemma 2.1 above).

LEMMA 3.1. *The best L^2 polynomial approximation to $(t-a)^{-1}$ with $a > 1$ over the interval $[-1, 1]$ is given by*

$$v_n(t) = -\sum_{k=0}^{n} (2k+1)\, Q_k(a)\, P_k(t)$$
$$= \frac{1}{t-a} - \frac{n+1}{t-a}\{P_{n+1}(t)\, Q_n(a) - P_n(t)\, Q_{n+1}(a)\},$$

where $P_n(t)$ is the Legendre polynomial of degree n and $Q_n(t)$ is the associated Legendre function.

Proof. It is known that the best L^2 polynomial approximation to a function $f \in L^2[-1, 1]$ is given by the partial sum of the Fourier series with respect to the normalized Legendre polynomials: that is, with respect to $\hat{P}_n(t) = (n + \frac{1}{2})^{1/2} P_n(t)$. We use A_k to denote the Fourier coefficient. Then

$$A_k = \int_{-1}^{1} \frac{\hat{P}_k(u)}{u-a}\, du$$

so that $v_n(t) = \sum_{k=0}^{n} A_k \hat{P}_k(t)$.

We recall ([10, p. 181, (34)]) that the associated Legendre function $Q_n(x)$ is related to the Legendre polynomial by

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\,dt.$$

Thus

$$A_k = \left(k+\frac{1}{2}\right)^{1/2} \int_{-1}^{1} \frac{P_k(u)}{u-a}\,du = -2\left(k+\frac{1}{2}\right)^{1/2} Q_k(a).$$

Hence

$$v_n(t) = -2\sum_{k=0}^{n} \left(k+\frac{1}{2}\right) Q_k(a)\,P_k(t).$$

In order to obtain the second formula, we use two known results.

(A) $\sum_{k=0}^{n}(2k+1)\,P_k(t)\,P_k(x)$
$= ((n+1)/(t-x))\{P_{n+1}(t)\,P_n(x) - P_n(t)\,P_{n+1}(x)\}$

(the Christoffel–Darboux formula—[10, p. 179, (10)]).

(B) $P_{n+1}(x)\,Q_n(x) - P_n(x)\,Q_{n+1}(x) = 1/(n+1)$ ([10, p. 172, (26)] with $\alpha = \beta = 0$).

Thus

$$\begin{aligned} v_n(t) &= \int_{-1}^{1} \sum_{k=0}^{n} \left(k+\frac{1}{2}\right) \frac{P_k(u)\,P_k(t)}{u-a}\,du \\ &= \frac{n+1}{2}\int_{-1}^{1} \frac{P_{n+1}(t)\,P_n(u) - P_n(t)\,P_{n+1}(u)}{(t-u)(u-a)}\,du \qquad \text{(using (A))} \\ &= \frac{n+1}{2(t-a)}\left\{P_{n+1}(t)\int_{-1}^{1}\left(\frac{1}{t-u}+\frac{1}{u-a}\right)P_n(u)\,du\right. \\ &\qquad \left. - P_n(t)\int_{-1}^{1}\left(\frac{1}{t-u}+\frac{1}{u-a}\right)P_{n+1}(u)\,du\right\} \\ &= \frac{n+1}{t-a}\{P_{n+1}(t)(Q_n(t)-Q_n(a)) - P_n(t)(Q_{n+1}(t)-Q_{n+1}(a))\} \end{aligned}$$

and the result follows by using (B).

LEMMA 3.2. *The error in the best L^2 approximation to $(t-a)^{-1}$ with $a>1$ over the interval $[-1, 1]$ is given by*

$$E_n = (n+1)\sqrt{2}(a^2-1)^{-1/2} \times \{Q_n(a) - \alpha Q_{n+1}(a)\}^{1/2}\{Q_n(a) - \alpha^{-1}Q_{n+1}(a)\}^{1/2},$$

where $\alpha = a - (a^2-1)^{1/2}$.

Proof. From the Parseval equation we obtain

$$(E_n)^2 = \int_{-1}^{1} (t-a)^{-2}\,dt - \sum_{k=0}^{n} (A_k)^2.$$

But

$$\sum_{k=0}^{n} (A_k)^2 = \int_{-1}^{1} \sum_{k=0}^{n} \frac{A_k \hat{P}_k(t)}{t-a}\,dt = \int_{-1}^{1} \frac{v_n(t)}{t-a}\,dt$$
$$= \int_{-1}^{1} (t-a)^{-2}\,dt + (n+1) \times \int_{-1}^{1} \frac{P_n(t)\,Q_{n+1}(a) - P_{n+1}(t)\,Q_n(a)}{(t-a)^2}\,dt,$$

from which it follows that

$$(E_n)^2 = 2(n+1)\{Q_n'(a)\,Q_{n+1}(a) - Q_{n+1}'(a)\,Q_n(a)\}.$$

We now use the recurrence relation for Q_n ([10, p. 179, (12); p. 181, note at top]),

$$(1-a^2)\,Q_n'(a) = n\{Q_{n-1}(a) - aQ_n(a)\}$$
$$= (n+1)\{aQ_n(a) - Q_{n+1}(a)\}$$

to obtain that

$$(E_n)^2 = \frac{2(n+1)^2}{a^2-1}\{(Q_n(a))^2 - 2aQ_n(a)\,Q_{n+1}(a) + (Q_{n+1}(a))^2\}$$
$$= \frac{2(n+1)^2}{a^2-1}\{Q_n(a) - \alpha Q_{n+1}(a)\}\{Q_n(a) - \alpha^{-1}Q_{n+1}(a)\}.$$

LEMMA 3.3. *For $a>1$,*

$$Q_n(a) = \frac{\alpha^{n+1/2}\pi^{1/2}}{2^{1/2}(a^2-1)^{1/4}} \cdot \frac{1}{n^{1/2}} \left\{1 + \frac{c(a)}{n} + O\left(\frac{1}{n^2}\right)\right\}$$

as $n \to \infty$, *where* $c(a)$ *is a constant depending only on* a *and* $\alpha = a - (a^2 - 1)^{1/2}$.

Proof. We shall make use of Watson's lemma ([23, p. 71; 27]) which asserts that if $q(t)$ is defined and real for $t \geqslant 0$ and satisfies

$$q(t) \sim \sum_{k=0}^{\infty} q_k t^{(k+i-j)/j}$$

as $t \to 0+$, where i and j are positive constants, then

$$\int_0^{\infty} e^{-xt} q(t)\, dt \sim \sum_{k=0}^{\infty} \Gamma\left(\frac{k+i}{j}\right) q_k x^{-(k+i)/j}$$

as $x \to \infty$ provided that the integral exists for all large x.

From [10, p. 181, (32)], we obtain that

$$Q_n(a) = \int_0^{\infty} \{a + (a^2 - 1)^{1/2} \cosh x\}^{-n-1}\, dx.$$

Set $a + (a^2 - 1)^{1/2} \cosh x = \alpha^{-1} e^t$. Thus

$$e^t\, dt = \alpha (a^2 - 1)^{1/2} \sinh x\, dx = dx/q(t),$$

where $q(t) = (e^{2t} - 2a\alpha e^t + \alpha^2)^{-1/2}$. Therefore

$$Q_n(a) = \alpha^{n+1} \int_0^{\infty} e^{-nt} q(t)\, dt.$$

Now $1 - 2a\alpha + \alpha^2 = 0$, and $e^{2t} - 2a\alpha e^t + \alpha^2 \sim 2t(1 - a\alpha)$ as $t \to 0+$. Hence

$$q(t) = (2t)^{-1/2} (1 - a\alpha)^{-1/2} \left(1 + \sum_{k=1}^{\infty} q_k t^k\right) \qquad \text{as} \quad t \to 0+.$$

The result now follows from Watson's lemma.

LEMMA 3.4. *In the notation of Lemma* 3.2,

$$E_n = \frac{\sqrt{\pi}}{(a^2 - 1)^{1/2}} \alpha^{n+1} \left(1 + O\left(\frac{1}{n}\right)\right) \qquad \textit{as} \quad n \to \infty.$$

Proof. From Lemma 3.2,

$$(E_n)^2 = \frac{2(n+1)^2}{a^2 - 1} \{Q_n(a) - \alpha Q_{n+1}(a)\}\{Q_n(a) - \alpha^{-1} Q_{n+1}(a)\}.$$

From Lemma 3.3,

$$Q_n(a)-\alpha^{-1}Q_{n+1}(a)=\frac{\alpha^{n+1/2}\pi^{1/2}}{2^{1/2}(a^2-1)^{1/4}}\{n^{-1/2}-(n+1)^{-1/2}+O(n^{-5/2})\}$$
$$=\frac{\alpha^{n+1/2}\pi^{1/2}}{2^{3/2}(a^2-1)^{1/4}}(n^{-3/2}+O(n^{-5/2})),$$

and

$$Q_n(a)-\alpha Q_{n+1}(a)=\frac{\alpha^{n+1/2}(1-\alpha^2)\,\pi^{1/2}}{2^{1/2}(a^2-1)^{1/4}}(n^{-1/2}+O(n^{-3/2})),$$

from which the result follows.

4. Solution to the Problem when $p=2$

We have

$$\varepsilon_n^{(2)}=\|\gamma_n(-t)-(1+t)^{-1}\|_2.$$

In order to minimize this expression as γ_n varies over the class of polynomials of degree n, we must find the best L^2 polynomial approximation to $(1+t)^{-1}$ over the interval $[0, 1]$. This problem we have solved in Section 3 above. Thus we can again calculate each row of the corresponding matrices C and B, and the minimal value for $\varepsilon_n^{(2)}$. Again it turns out that these matrices are particularly nice and can be determined explicitly. They are again regular summability methods. We shall also investigate (in Section 5) the asymptotic behaviour of the matrix B and find the Gaussian distribution to which it is asymptotic.

We require then the best L^2 polynomial approximation to $(1+t)^{-1}$ over the interval $[0, 1]$. Let

$$\gamma_n(t)=-2v_n(1+2t).$$

Thus

$$\int_0^1|\gamma_n(-t)-(1+t)^{-1}|^2\,dt=2\int_{-1}^1|v_n(x)-(x-3)^{-1}|^2\,dx.$$

So we may apply the results of Section 3 to obtain that

$$\gamma_n(t)=-\left(\frac{n+1}{t-1}\right)\left\{\frac{1}{n+1}+P_n(1+2t)\,Q_{n+1}(3)-P_{n+1}(1+2t)\,Q_n(3)\right\}.$$

From Rodrigues' formula, we obtain that

$$\begin{aligned}
P_n(1+2t) &= \frac{1}{n!}\left(\frac{d}{dt}\right)^n t^n(1+t)^n = \sum_{k=0}^{n}\binom{n}{k}^2 t^{n-k}(1+t)^k \\
&= \sum_{j=0}^{n} t^{n-j}\sum_{k=j}^{n}\binom{n}{k}^2\binom{k}{j} = \sum_{j=0}^{n} t^j \sum_{k=0}^{j}\binom{n}{k}^2\binom{n-k}{n-j} \\
&= \sum_{j=1}^{n}\binom{n}{j}\binom{n+j}{j}t^j.
\end{aligned}$$

Thus we can obtain γ_n explicitly.

We also have, from Lemmas 3.2 and 3.4,

$$\begin{aligned}
\varepsilon_n^{(2)}(C) &= \frac{\lambda\sqrt{\pi}}{2}\lambda^n\left(1+O\left(\frac{1}{n}\right)\right) \qquad (\lambda = 3-2\sqrt{2}) \\
&\sim \frac{\lambda\sqrt{\pi}}{2}\lambda^n \qquad \text{as } n\to\infty.
\end{aligned}$$

We recall that $\varepsilon_n^{(\infty)} \sim \lambda^n/4$ and, observing that $\lambda\sqrt{\pi} \doteqdot 0.152$, we see that there is a small (and insignificant with respect to the exponential, λ^n) reduction in the constant.

Again we shall obtain the rows of the more convenient sequence-to-sequence matrix B.

Recalling that $t\beta_n(t) = \gamma_n(0) + (t-1)\gamma_n(t)$ and observing that $\gamma_n(0) = 1 + (n+1)\{Q_{n+1}(3) - Q_n(3)\}$, it follows that

$$\begin{aligned}
\beta_n(t) &= (n+1)\sum_{k=1}^{n}\left\{\binom{n+1}{k+1}\binom{n+k+2}{k+1}Q_n(3)\right. \\
&\qquad \left. - \binom{n}{k+1}\binom{n+k+1}{k+1}Q_{n+1}(3)\right\}t^k \\
&= \sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+k+1}{k+1}\{n(Q_n(3)-Q_{n+1}(3)) \\
&\qquad + k(Q_n(3)+Q_{n+1}(3)) + 2Q_n(3)\}t^k \\
&= \sum_{k=0}^{n} b_{n,k}t^k,
\end{aligned}$$

say. From the formula

$$Q_n(3) = \int_0^\infty \{3+2\sqrt{2}\cosh t\}^{-n-1}\,dt$$

it is easy to see that $\{Q_n(3)\}$ is a decreasing sequence. Thus the matrix B is a positive triangle. The sum of the row of order n is $\gamma_n(0)$, and we have

$$\begin{aligned}\gamma_n(0) &= 1-(n+1)\{Q_n(3)-Q_{n+1}(3)\}\\ &\to 1 \qquad \text{as } n\to\infty.\end{aligned}$$

Further, an application of Stirling's formula yields that $\lim_{n\to\infty} b_{n,k}=0$ for each k.

It again follows from Toeplitz's theorem that B is a regular matrix.

5. Asymptotic Expansions of the Best Matrices

In this section we show that both the best L^∞ matrix and the best L^2 matrix obtained above give rise to Gaussian distributions of the form

$$b_{n,k}=\sqrt{\frac{c}{n\pi}}\exp(-\text{ch}^2/n)\left\{1+O\left(\frac{1}{n}\right)+O\left(\frac{h^3}{n^2}\right)\right\}$$

for $|h|\leqslant\theta n$ with some $\theta\in(0,1)$, and large n, where

$$k=k_n+h=dn+\delta+h, \qquad 0\leqslant k\leqslant n.$$

We call c, d and δ the *parameters* of the asymptotic which are to be restricted as follows:

$$c>0, \qquad 0<d<1, \qquad \delta \text{ real}.$$

Compare, for example [13, Sections 9.1 to 9.3].

For this section we adopt the following notation. The best L^∞ matrix can be written as

$$b_{n,k}^{(\infty)}=\frac{\lambda^n 2^{2k-2}(n+k-1)!}{(n-k)!\,(2k+2)!}\lambda_{n,k}^{(\infty)},$$

where

$$\begin{aligned}\lambda_{n,k}^{(\infty)} &= 4\sqrt{2}(\sqrt{2}n+1)k^2+2(4\sqrt{2}n^2+9n+2\sqrt{2})k\\ &\quad+2n(2n^2+4\sqrt{2}n+3).\end{aligned}$$

And the best L^2 matrix can be written as

$$b_{n,k}^{(2)}=\binom{n+1}{k+1}\binom{n+k+1}{k+1}\lambda_{n,k}^{(2)},$$

where

$$\lambda_{n,k}^{(2)} = n(Q_n(3) - Q_{n+1}(3)) + k(Q_n(3) + Q_{n+1}(3)) + 2Q_n(3).$$

In order to find the asymptotics, it is convenient first to set $k = n(1+t)/\sqrt{2}$, where $|t|$ is relatively small. We use Stirling's formula in the form

$$\log \Gamma(\alpha n + \beta) = \frac{1}{2}\log 2\pi - \alpha n + \left(\alpha n + \beta - \frac{1}{2}\right)\log(\alpha n) + O\left(\frac{1}{n}\right).$$

First we shall consider the L^∞ case. We have that

$$\begin{aligned}
&\log(n+k-1)!\\
&= \log \Gamma\left(n\left(1 + \frac{1}{\sqrt{2}}\right) + \frac{nt}{\sqrt{2}}\right)\\
&= \frac{1}{2}\log 2\pi - \left\{n\left(1+\frac{1}{\sqrt{2}}\right) + \frac{nt}{\sqrt{2}}\right\} + \left\{n\left(1+\frac{1}{\sqrt{2}}\right) - \frac{1}{2} + \frac{nt}{\sqrt{2}}\right\}\\
&\quad\times\left\{\log n + \log\left(1+\frac{1}{\sqrt{2}}\right) + \frac{t}{1+\sqrt{2}} - \frac{t^2}{2(1+\sqrt{2})^2} + O(t^3)\right\}\\
&\quad + O\left(\frac{1}{n}\right).
\end{aligned}$$

We can compute similar expressions for $\log(n-k)!$ and $\log(2k+2)!$. Collecting then yields

$$\begin{aligned}
&-\frac{1}{2}\log \pi - \frac{7}{2}\log n - n\log\lambda - n\sqrt{2}\log 2 - \frac{5}{4}\log 2\\
&\qquad + t\left(-\sqrt{2}\log 2 - \frac{3}{2}\right) + t^2\left(-\sqrt{2}n + \frac{11}{4}\right) + O(nt^3) + O\left(\frac{1}{n}\right).
\end{aligned}$$

We further have

$$\log \lambda_{n,k}^{(\infty)} = 3\log n + 4\log 2 + t - \frac{t^2}{4} + O(nt^3) + O\left(\frac{1}{n}\right)$$

and

$$\log \lambda^n 2^{2k-2} = n\log\lambda - 2\log 2 + n\sqrt{2}(1+t)\log 2.$$

Thus we obtain that

$$\begin{aligned}\log b_{n,k}^{(\infty)} &= \log \frac{2^{3/4}}{\sqrt{n\pi}} - \frac{t}{2} + t^2\left(-\sqrt{2}n + \frac{5}{2}\right) + O(nt^3) + O\left(\frac{1}{n}\right)\\ &= \log \frac{2^{3/4}}{\sqrt{n\pi}} + \left(\frac{5}{2} - \sqrt{2}n\right)\left\{t - \frac{1}{4(5/2 - \sqrt{2}n)}\right\}^2\\ &\quad + O(nt^3) + O\left(\frac{1}{n}\right)\\ &= \log \frac{2^{3/4}}{\sqrt{n\pi}} - 2\sqrt{n}\,\tau^2 + O(n\tau^3) + O\left(\frac{1}{n}\right),\end{aligned}$$

where $\tau = t + 1/4\sqrt{2}\,n$. Hence the parameters for the L^∞ asymptotic are

$$c = 2^{3/2}; \qquad d = \frac{1}{\sqrt{2}}; \qquad \delta = -\frac{1}{8}.$$

In the L^2 case, we proceed in a similar way and obtain that

$$\begin{aligned}\log \lambda_{n,k}^{(2)} &= \frac{1}{2}\log \pi + \frac{3}{4}\log 2 + \frac{1}{2}\log n + (n+1)\log \lambda\\ &\quad + \frac{t}{2} - \frac{t^2}{8} + O(nt^3) + O\left(\frac{1}{n}\right)\end{aligned}$$

and that

$$\begin{aligned}&\log \binom{n+1}{k+1}\binom{n+k+1}{k+1}\\ &\quad = -\log \pi - \log n - (n+1)\log \lambda + t(2\sqrt{2} - 4)\\ &\quad\quad + t^2(-\sqrt{2}n + 2\sqrt{2}) + O(nt^3) + O\left(\frac{1}{n}\right)\end{aligned}$$

so that

$$\begin{aligned}\log b_{n,k}^{(2)} &= \log \frac{2^{3/4}}{\sqrt{n\pi}} + t\left(2\sqrt{2} - \frac{7}{2}\right) + t^2\left(-\sqrt{2}n + 2\sqrt{2} - \frac{1}{8}\right)\\ &\quad + O(nt^3) + O\left(\frac{1}{n}\right)\\ &= \log \frac{2^{3/4}}{\sqrt{n\pi}} - \sqrt{2}n\,\tau^2 + O(n\tau^3) + O\left(\frac{1}{n}\right),\end{aligned}$$

where $\tau = t + (4\sqrt{2} - 7)/4\sqrt{2}n$. Hence the parameters for the L^2 asymptotic are

$$c = 2^{3/2}; \qquad d = \frac{1}{\sqrt{2}}; \qquad \delta = -\left(\frac{4\sqrt{2} - 7}{8}\right) \doteqdot -\frac{1}{5.96}.$$

We see that the asymptotics differ only in the parameter δ. The maximum position for the L^2 distribution occurs slightly to the left of the maximum for the L^∞ distribution.

6. Comparison with Known Matrices

There are two matrices occurring in the literature, which are effective for accelerating the convergence of convergent alternating series. One is the Euler matrix, (E, q). (See, for example [13, p. 178].) The matrix $(E, 1)$ is widely used in numerical analysis. The other matrix is called the Čebyšev matrix and was introduced by Bilodeau [4]. In this section we compute the L^∞ errors for these matrices and their corresponding asymptotic parameters.

The Euler matrix (E, q) is generated by

$$\beta_n(t) = \left(\frac{q + t}{q + 1}\right)^n \qquad (q > 0)$$

so that

$$b_{n,k} = q^{n-k}(1 + q)^{-n} \binom{n}{k} \qquad (0 \leqslant k \leqslant n).$$

From the relation $(1 - t)\,\gamma_n(t) = \beta_n(1) - t\beta_n(t)$, we obtain that

$$\varepsilon_n^{(\infty)} = \max_{0 \leqslant t \leqslant 1} |\gamma_n(-t) - (1 + t)^{-1}| = \max_{0 \leqslant t \leqslant 1} \left|\frac{t\beta_n(-t) + \beta_n(1) - 1}{1 + t}\right|.$$

For the (E, q) matrix, we then have

$$\varepsilon_n^{(\infty)}(E, q) = \max_{0 \leqslant t \leqslant 1} \frac{t\,|q - t|^n}{(1 + t)(1 + q)^n}.$$

Let $m_n(t) = t(q - t)^n(1 + t)^{-1}$ for $0 \leqslant t \leqslant 1$. The critical points for $m_n(t)$ are $t = 0$, $t = 1$, $t = q$ and $t = t_n$, where $2nt_n = \{1 + 2n(1 + 2q) + n^2\}^{1/2} - 1 - n$. Thus

$$\varepsilon_n^{(\infty)}(E, q) = \max\left\{\frac{|q - 1|^n}{2(1 + q)^n}, \frac{t_n\,|q - t_n|^n}{(1 + t_n)(1 + q)^n}\right\}.$$

But $t_n = q/n + O(1/n^2)$ so that

$$\frac{m_n(t_n)}{(1+q)^n} = \frac{q^{n+1}}{n+q}\frac{(1-1/n)^n}{(1+q)^n}\left\{1+O\left(\frac{1}{n}\right)\right\}$$
$$= \frac{q}{ne}\left(\frac{q}{1+q}\right)^n\left\{1+O\left(\frac{1}{n}\right)\right\} \qquad \text{as } n\to\infty.$$

Thus, for large n,

$$\varepsilon_n^{(\infty)}(E,q) = \frac{(1-q)^n}{2(1+q)^n} \qquad \text{for } 0<q\leqslant 1/2$$

and

$$\varepsilon_n^{(\infty)}(E,q) = \frac{t_n\,|q-t_n|^n}{(1+t_n)(1+q)^n} \qquad \text{for } q>1/2.$$

This immediately gives that $(E, 1/2)$ is the "best" Euler method and that the minimal error is

$$\varepsilon_n^{(\infty)}(E,\tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{1}{3})^n.$$

To find the asymptotic expansion, it is convenient to first set $k = n(1+t)/(1+q)$. Then, using Stirling's formula we obtain that

$$\log b_{n,k} = \log\frac{1+q}{\sqrt{2q\pi n}} + \frac{t(1-q)}{2q} + t^2\left\{\frac{-n}{2q}+\frac{1+q^2}{4q^2}\right\}$$
$$+\,O(nt^3) + O\left(\frac{1}{n}\right)$$
$$= \log\frac{1+q}{\sqrt{2q\pi n}} - \frac{n}{2q}\tau^2 + O(n\tau^3) + O\left(\frac{1}{n}\right),$$

where $\tau = t-(1-q)/2n$. Hence the parameters for the (E, q) asymptotic are (in the notation of Section 5 above)

$$c = \frac{(1+q)^2}{2q}: \qquad d = \frac{1}{2q}: \qquad \delta = \frac{1-q}{2q}.$$

Compare Theorem 138 in [13].

The Čebyšev matrix T is generated by

$$\beta_n(t) = \frac{T_n(1+2t)}{T_n(3)}$$

so that

$$b_{n,k} = \frac{2^{2k}n(n+k-1)!}{(n-k)!\,2k!\,T_n(3)}.$$

Thus

$$\varepsilon_n^{(\infty)}(T) = \max_{0 \leqslant t \leqslant 1} \left| \frac{tT_n(1-2t)}{(1+t)\,T_n(3)} \right|.$$

First we note that

$$\varepsilon_n^{(\infty)}(T) \geqslant \frac{|T_n(-1)|}{2T_n(3)} = \frac{1}{2T_n(3)} \qquad \text{since } T_n(-1) = (-1)^n.$$

Further

$$\varepsilon_n^{(\infty)}(T) \leqslant \frac{\max_{0 \leqslant t \leqslant 1} |T_n(1-2t)|}{2T_n(3)} = \frac{\max_{-1 \leqslant x \leqslant 1} |T_n(x)|}{2T_n(3)} = \frac{1}{2T_n(3)}.$$

Thus

$$\varepsilon_n^{(\infty)}(T) = \frac{1}{2T_n(3)} = \frac{1}{\lambda^n + \lambda^{-n}} \sim \lambda^n \qquad \text{as } n \to \infty.$$

Comparing this error with the error for the best L^∞ matrix, we see that it only lacks the factor 1/4. Therefore T is an excellent matrix for applications, and indeed is much better than any Euler matrix.

To find the asymptotic for T, it is again convenient to first set $k = n(1+t)/\sqrt{2}$. Then, using Stirling's formula, we obtain that

$$\begin{aligned}\log b_{n,k} &= \frac{2^{3/4}}{\sqrt{n\pi}} + \frac{t}{2} + t^2\left(-n\sqrt{2} + \frac{7}{4}\right) + O(nt^3) + O\left(\frac{1}{n}\right)\\ &= \log \frac{2^{3/4}}{\sqrt{n\pi}} - n\sqrt{2}\,\tau^3 + O(n\tau^3) + O\left(\frac{1}{n}\right),\end{aligned}$$

where $\tau = t - 1/4\sqrt{2}n$. Hence the parameters for the T asymptotic are

$$c = 2^{3/2}: \qquad d = 1/\sqrt{2}: \qquad \delta = 1/8.$$

It is interesting to note that these parameters differ from those for the best L^∞ matrix and for the best L^2 matrix only in the parameter δ. Note that here $\delta = 1/8$ whereas for the best L matrix, $\delta = -1/8$.

7. Truncated Euler Methods

We first note that the efficiency of the best methods obtained above is not controlled by their Gaussian distributions, since we can imitate such a distribution for an Euler method, (E, q) with $q = 1/\sqrt{2}$. This gives

$$\varepsilon_n^\infty\left(E, \frac{1}{\sqrt{2}}\right) = \frac{1}{ne\sqrt{2}}\left(\frac{1}{1+\sqrt{2}}\right)^n\left(1 + O\left(\frac{1}{n}\right)\right)$$

and

$$\frac{1}{1+\sqrt{2}}=\sqrt{2}-1 \doteqdot 0.41421>\lambda=3-2\sqrt{2} \doteqdot 0.17157.$$

We show below that we can improve the error for Euler methods by truncating the matrix. However, the best truncated Euler method is not as good, asymptotically, as the Best methods obtained above.

Let $n=\mu+\nu$ with $\mu=\theta\nu$, so that $n=(1+\theta)\nu$. Thus

$$\sum_{k=0}^{n}(-t)^k=\frac{1-(-t)^{\mu+\nu+1}}{1+t}$$

and the (E,q)-transform of order ν is then

$$\begin{aligned}\left(\frac{q}{1+q}\right)^{\nu}\sum_{k=0}^{\nu}\binom{\nu}{k}q^{-k}\left(\frac{1-(-t)^{\mu+k+1}}{1+t}\right)\\ =\frac{1}{1+t}\left\{1-(-t)^{\mu+1}\left(\frac{q-1}{q+1}\right)^{\nu}\right\}.\end{aligned}$$

Thus

$$\varepsilon_n^{(\infty)}=\max_{0\leqslant t\leqslant 1}\left|\frac{t}{1+t}t^{\mu}\left(\frac{q-t}{q+1}\right)^{\nu}\right|.$$

For simplicity, we consider the modified error given by

$$\eta=\eta_n^{(\infty)}=\max_{0\leqslant t\leqslant 1}\left|t^{\mu}\left(\frac{q-t}{q+1}\right)^{\nu}\right|=\max_{0\leqslant t\leqslant 1}\left|\left(\frac{t^{\theta}(q-t)}{q+1}\right)^{1/(1+\theta)}\right|^{n}.$$

First we note that if $\theta=0$, then

$$\begin{aligned}(\eta)^{1/n}&=\max\left\{\frac{q}{q+1},\frac{|1-q|}{q+1}\right\}\\ &=\frac{1-q}{1+q}\qquad \text{if } 0<q\leqslant\frac{1}{2}\\ &=\frac{q}{1+q}\qquad \text{if } q\geqslant\frac{1}{2}.\end{aligned}$$

These are consistent with results in Section 6 above.

From now on, we assume that $\mu\geqslant 1$ so that $\theta>0$. Let $u(t)=t^{\theta}(q-t)$ $(t\geqslant 0)$. Thus $u'(t)=0$ if and only if $t=\theta q/(1+\theta)$. Therefore, there are three cases to consider:

(A) $q > 1$ with $1 \leqslant \theta q/(1+\theta)$,

(B) $\theta q/(1+\theta) < 1 \leqslant q$,

(C) $0 < q < 1$.

Case A. Let $y(t) = 2t^{\theta}(q-t) - (q+1)(1-t)$. Since $\theta q - 1 - \theta \geqslant 0$, $y'(t) = 2t^{\theta-1}(\theta q - t - \theta t) + q + 1$, and $y(1) = 2(q-1) > 0$, it follows that $y(t) > 0$ for $0 \leqslant t \leqslant 1$. Thus $\eta_n^{(\infty)}(E, q) \geqslant \eta_n^{(\infty)}(E, 1)$ and $\varepsilon_n^{(\infty)}(E, q) \geqslant \varepsilon_n^{(\infty)}(E, 1)$. Therefore this case is of no interest.

Case B. Here we have

$$(\eta)^{1/n} = \left(\frac{\theta^{\theta}}{1+q}\right)^{1/(1+\theta)} \cdot \frac{q}{1+\theta} = v(\theta),$$

say. Now

$$\begin{aligned} \frac{v'(\theta)}{v(\theta)} &= \frac{\log(\theta(1+q))}{(1+\theta)^2} \\ &= 0 \qquad \text{if and only if } \theta = \frac{1}{1+q} \left(\leqslant \frac{1}{2} \text{ since } q \geqslant 1\right). \end{aligned}$$

Thus the minimal value of $(\eta)^{1/n}$ is given by

$$\frac{1-\theta}{1+\theta} = \frac{q}{2+q}$$

from which we obtain the best result from $\theta = \frac{1}{2}$ and $q = 1$ yielding

$$\eta_n^{(\infty)} = (\tfrac{1}{3})^n.$$

This improves on the result $\eta_n^{(\infty)} = (\frac{1}{2})^n$ with $\theta = 0$ and $q = 1$.

Case C. Here we have

$$\begin{aligned} (\eta)^{1/n} &= \max\left\{\left(\frac{\theta^{\theta}}{1+q}\right)^{1/(1+\theta)} \cdot \frac{q}{1+\theta}, \left(\frac{1-q}{1+q}\right)^{1/(1+\theta)}\right\} \\ &= \max\{v(\theta), w(\theta)\}, \end{aligned}$$

say. Since

$$\frac{w'(\theta)}{w(\theta)} = \frac{-\log((1-q)/(1+q))}{(1+\theta)^2} > 0$$

and

$$\frac{v'(\theta)}{v(\theta)} = \frac{\log(\theta(1+q))}{(1+\theta)^2},$$

we obtain that $w(\theta)$ increases from $(1-q)/(1+q)$ to 1 as θ increases from 0 to ∞, and that $v(\theta)$ decreases from $q/(1+q)$ to $q/(2+q)$ as θ increases from 0 to $1/(1+q)$, and then increases to q as θ increases to ∞. We also have that $w(\theta) = v(\theta)$ if and only if

$$1 - q = \theta^\theta \left(\frac{q}{1+\theta}\right)^{1+\theta} = x(\theta), \qquad (*)$$

say. Let $x(0) = q$ (by continuity) and note that

$$\frac{x'(\theta)}{x(\theta)} = \log\left(\frac{\theta q}{1+\theta}\right) < 0.$$

Thus $x(\theta)$ decreases from q to 0 as θ increases from 0 to ∞.

For $0 < q \leqslant \frac{1}{2}$, we now obtain that

$$(\eta)^{1/n} = \left(\frac{1-q}{1+q}\right)^{1/(1+\theta)}$$

with the best value at $\theta = 0$. Thus there is no improvement in the error.

For $\frac{1}{2} \leqslant q < 1$, there is a unique solution of (*), which we call ϕ. Thus

$$\begin{aligned}(\eta)^{1/n} &= v(\theta) \qquad \text{for } 0 < \theta \leqslant \phi \\ &= w(\theta) \qquad \text{for } \phi \leqslant \theta.\end{aligned}$$

The minimal value (for fixed q) is then given by

$$(\eta)^{1/n} = v(\phi) = w(\phi).$$

To find this, we must solve the transcendental equation

$$1 - q = \phi^\phi \left(\frac{q}{1+\phi}\right)^{1/(1+\phi)},$$

which unfortunately has no explicit solution. Standard numerical techniques yield that the optimal value of q is approximately 0.70297, with $\phi \doteqdot 0.330017$ and $(\eta)^{1/n} \doteqdot 0.269015$. This is the best value that can be obtained using an Euler method. We summarise the results in Table I.

TABLE I

Method		$(\eta)^{1/n}$
"Popular"	$(E, 1)$	0.5
"Best"	$(E, \frac{1}{2})$	0.333333
Truncated	$(E, 1)$	0.333333
Best truncated	(E, q)	0.269015
Best L^{∞}		0.171573

8. New Effective Methods

It appears that the effectiveness of a method is obtained from the appearance of binomial coefficients of the form $\binom{n+k}{n-k}$. We therefore suggest for computational purposes the Best-L^1 matrix obtained by Fiedler and Jurkat [11] and the matrix given by

$$\begin{aligned} b_{n,k} &= 4^k \binom{n+k}{n-k}, && 0 \leqslant k \leqslant n \\ &= 0, && \text{otherwise.} \end{aligned}$$

We must calculate

$$\beta_n(x) = \sum_{k=0}^{n} 4^k \binom{n+k}{n-k} x^k.$$

Now

$$T_n(1+2x) = \sum_{k=0}^{n} 4^k \frac{n}{n+k} \binom{n+k}{n-k} x^k.$$

By considering $x^n T_n(1+2x)$ and differentiating with respect to x, we obtain that

$$\begin{aligned} \beta_n(x) &= T_n(1+2x) + \frac{2x}{n} T_n'(1+2x) \\ &= T_n(1+2x) + 2xU_{n-1}(1+2x) \\ &= U_n(1+2x) - U_{n-1}(1+2x), \end{aligned}$$

where $U_n(t)$ is the Čebyšev polynomial of the second kind. See, for example [10, Section 10.11, pp. 183–187]. Thus

$$\begin{aligned}\beta_n(1) &= U_n(3) - U_{n-1}(3)\\ &= \frac{T_{n+2}(3) - 4T_{n+1}(3) + 3T_n(3)}{8} \sim \frac{\lambda^{-n}}{1+\lambda} \qquad \text{as } n \to \infty.\end{aligned}$$

Therefore, in order that the method be regular, we must now redefine the matrix to be

$$\begin{aligned} b_{n,k} &= (1+\lambda)\,\lambda^n 4^k \binom{n+k}{n-k}, && 0 \leqslant k \leqslant n\\ &= 0, && \text{otherwise}\end{aligned}$$

and now

$$\beta_n(x) = (1+\lambda)\,\lambda^n\{U_n(1+2x) - U_{n-1}(1+2x)\}.$$

We shall call this method U. We must investigate the behaviour of

$$\psi_n(t) = U_n(t) - U_{n-1}(t)$$

for $t \in [-1, 1]$, or equivalently, $\psi_n(\cos\theta)$ for $\theta \in [0, \pi]$. Now

$$\psi_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos\theta/2}.$$

But $\cos(n+\frac{1}{2})\theta = 0$ if and only if $\theta = \pi(1+2j)/(1+2n)$ $(0 \leqslant j \leqslant n)$. There is thus only one zero (at $\theta = \pi$) in common with the zeros of $\cos\theta/2$. Also $\lim_{\theta\to\pi-} \psi_n(\cos\theta) = (-1)^n(2n+1)$.

For $0 \leqslant \theta \leqslant 2n\pi/(1+2n)$, we observe that $\cos\theta/2$ decreases to

$$\cos\frac{2n\pi}{1+2n} = \sin\frac{\pi}{2(1+2n)} \geqslant \frac{1}{2n+1}$$

(by Jordan's inequality) so that

$$|\psi_n(\cos\theta)| \leqslant 2n+1.$$

For $2n\pi/(1+2n) \leqslant \theta \leqslant \pi$, we set $\phi = \pi - \theta$, observe that

$$\psi_n(\cos\theta) = (-1)^n \frac{\sin(n+1/2)\phi}{\sin\phi/2},$$

and obtain that

$$|\psi_n(\cos\theta)| \leqslant \frac{\sin(n+1/2)(\pi/(2n+1))}{\sin \pi/2(2n+1)}$$

$$= \frac{\sin \pi/2}{\sin \pi/2(2n+1)} \leqslant 2n+1.$$

Thus

$$\max_{0\leqslant x\leqslant 1} \left| \frac{x}{1+x} \psi_n(1-2x) \right| = \varepsilon_n^{(\infty)}(U) \leqslant \left(n+\frac{1}{2}\right)(1+\lambda)\lambda^n \qquad \text{as } n\to\infty.$$

Following the procedure of Section 5, we obtain that

$$\log b_{n,k} = \log \frac{2^{3/4}}{\sqrt{n\pi}} + t\left(\sqrt{2}-\frac{1}{2}\right) + t^2\left(-\sqrt{2}n + \sqrt{2} + \frac{1}{4}\right)$$

$$+ O(nt^3) + O\left(\frac{1}{n}\right)$$

$$= \log \frac{2^{3/4}}{\sqrt{n\pi}} - \sqrt{2}n\,\tau^2 + O(n\tau^3) + O\left(\frac{1}{n}\right),$$

where

$$\tau = t + \frac{\sqrt{2}-1/2}{2\sqrt{2}n}.$$

It now follows that the parameters for the U-asymptotics are

$$c = 2^{3/2}; \qquad d = \frac{1}{\sqrt{2}}; \qquad \delta = -\left(\frac{2\sqrt{2}-1}{8}\right) \doteqdot \frac{-1}{4.4}.$$

We see that these parameters differ from the Best parameters only in δ. The maximum for U occurs slightly to the right of the maximum for the L^∞ case.

9. Concluding Remarks

In the cases of Best matrices obtained here, we have given an explicit form for the entries in the matrices. It is remarkable that these matrices should be so "nice"—they are in fact positive regular triangles. The individual entries are very "nice" too. In the L^∞ case, the entries can be expressed in terms of rational numbers and $\sqrt{2}$: in the L^2 case, in terms of rational numbers and log 2. This latter result follows from the formula (see [10, p. 181, (35); 1, p. 334, 8.6.19]).

$$Q_n(x) = Q_0(x)\,P_n(x) - \sum_{k=1}^{[(n+1)/2]} \frac{2n-4k+3}{(2k-1)(n+1-k)} P_{n-2k+1}(x)$$

and the fact that

$$Q_0(x) = \frac{1}{2} \log \frac{x+1}{x-1}.$$

For applications to numerical calculation, we observe that for "smaller" values of n, the Best methods will give the best results. For example, for the log 2 series, the Best L^∞ method gives an error of about 1.5×10^{-9} for nine terms of the series. This compares with an error of about 4.5×10^{-6} using the $(E, \frac{1}{2})$ method and about 3.7×10^{-5} for the $(E, 1)$ method. Thus the Best methods have considerable practical advantages for computational purposes.

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